LIKE-WITH-LIKE PREFERENCE AND SEXUAL MIXING MODELS

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ABSTRACT

We present two new general methods for incorporating like-with-like preference into one-sex mixing models in epidemiology. The first is a generalization of the Sattenspiel mixing equation, while the second comprises a transformation of a general preference function for partners of similar sexual activity levels. Both methods satisfy the constraints implicit in a mixing model. We then illustrate how the transformation preference method behaves and compare it with the standard proportionate mixing model.

INTRODUCTION

In models of the dynamics of sexually transmitted diseases (STDs) within populations with heterogeneous sexual activity, it is necessary to specify the contact preference (who mixes with whom). Thus, for each level of sexual activity (number of new partners per unit time) we must know the fraction of partners coming from all other levels of activity. For practical modeling purposes, we require some function of activity which both makes analysis straightforward and is a reasonably accurate characterization of observed mixing patterns. Until recently the proportionate mixing model- equation (1) below-was the only description of the mixing process available in analytic form, although arbitrary rules may, of course, be applied in stochastic simulations of the interaction of

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individuals. Proportionate mixing has been used extensively in a variety of situations by Barbour (1978), Nold (1980), Anderson and May (1984), Dietz and Schenzle (1985), Anderson and Grenfell (1986), Hethcote and Van Ark (1987), Castillo-Chavez et al. (1988, and in press). While this model has also proved useful in the study of the epidemiology of STDs (see Hethcote and Yorke, 1984, for an outstanding example), it has become clear in recent years that more realistic mixing models are required for any detailed understanding of the transmission dynamics of HIV-1 and for any study of the possible value of various control measures.

At present there are few robust data on contact preference in a given community, although estimates of the distribution of *numbers* of sexual partners have been derived from various surveys. In the absence of detailed mixing information, models of HIV-1 transmission must take into account as many mixing patterns as possible in order that the impact of any preferential mixing on the AIDS epidemic in a given population be better understood.

Recently Sattenspiel (1987a and 1987b) questioned the use of proportionate mixing in dynamic models for the spread of diseases in structured populations. She emphasizes in those diseases for which the geographic and social structure plays an important role. Since then her ideas have been expanded by her and her collaborators (see Sattenspiel and Simon 1988, Sattenspiel et al. (ms.), and Jacquez et al.(in press)). Sattenspiel et al. (ms.) and Jaquez et al. (in press) have presented a new mixing model which has a restricted form of like-with-like preference (individuals have a bias towards others of the same activity level) superimposed upon a proportionate mixing background. Stanley and Hyman (in press) have examined some approximations to like-with-like mixing, and Stanley (personal communication) has developed a model where the preference of half of the population may be specified by the modeler, with the other half being defined by the preferences of the first. All of these approaches represent important and valuable contributions to the study of STD epidemiology.

In this paper we seek to extend the range of mixing models available to the modeler by introducing two new forms which satisfy the necessary contraints: (1) generalized Sattenspiel mixing, and (2) neighborhood mixing.

MIXING FUNCTIONS

In any one-sex model with heterogeneous sexual activity we have the mixing function $\rho(s, r)$, which specifies the fraction of s partners among individuals with activity r. There are three constraints which $\rho(s, r)$ must satisfy for all s and r:

(i) $\rho\,(s, r) \geq 0$,

(ii) $\int_0^\infty \rho\,(s, r)\,dr = \int_0^\infty \rho\,(r, s)\,ds = 1,$

(iii) $\rho\,(s, r)\, s\, N(s) = \rho\,(r, s)\, r\, N(r)$,

where N(x) is the number of people in the population with activity x (this is of course a function of time. However, we have suppressed time notation as (i - iii) must be true at *all* times. Conditions (i) and (ii) arise because $\rho(s,r)$ is in effect a probability density function, while condition (iii) expresses the requirement that the total number of partnerships of s-people with r-people must equal the total number of partnerships of r-people with s-people. These constraints are simple and obvious, but it is extraordinarily difficult to find functional forms for $\rho(s,r)$ which satisfy them simultaneously for all s,r, and time t.

We express the standard mixing model for proportionate mixing as

$$\rho\,(s, r) \;=\; \frac{r\,N(r)}{\displaystyle\int_0^\infty u\,N(u)\,du} \;. \qquad (1)$$

Here $\rho(s,r)$ is actually independent of s, and may be interpreted as saying that the fraction of partners taken by any individual in the population from individuals with activity r is proportionate to the total number of partnerships formed by all r-people, and clearly satisfies (i)-(iii).

The Sattenspiel mixing function is an extension of equation (1) to include a preference of individuals for partners with exactly the same activity level. In the continuous variables r and s used here, her function is

$$\rho\,(s, r) \;=\; (1 - \alpha)\,\frac{r\,N(r)}{\displaystyle\int_0^\infty u\,N(u)\,du} \;+\; \alpha\;\delta(s - r)\;, \qquad (2)$$

where $\delta(s\text{-}r)$ is a Dirac delta function and the constant α represents the bias towards partners of exactly the same activity. Although very useful for modeling purposes, and sufficient to demonstrate that even a small bias towards like-with-like can have a profound effect on epidemiological patterns, equation (2) is rather restricted as a general model of preference.

A more general alternative to proportionate mixing has been derived by Stanley (personal communication), and takes the form

$$\rho(s,r) = \begin{cases} \rho(r,s)\,\dfrac{rN(r)}{s\,N(s)}, & r < s \\[2ex] \dfrac{f(s,r)\,r\,N(r)}{\displaystyle\int_0^s f(s,u)\,u\,N(u)\,du}\left(1 - \displaystyle\int_0^s \rho(s,u)\,du\right), & r > s \end{cases} \tag{3}$$

where $\rho(r,s)$ for $r < s$ is arbitrarily specified by the modeler to suit available data, and the rest of the values are derived from this constraint. The function $f(s,r)$ appears to be arbitrary, and may be used to fine-tune the behaviour of $\rho(s,r)$ to the modeler's needs. It may be shown that equation (3) satisfies (i) - (iii). This general mixing function is potentially of great value in modeling studies.

We now introduce two new mixing functions which satisfy the constraints (i) - (iii).

GENERALIZED SATTENSPIEL MIXING

The first mixing model is a direct generalization of Sattenspiel's additive equation (2) to allow preferences for partners with activities which are arbitrary multiples of one's own. This takes the form

$$\rho(s,r) = \frac{r\,N(r) - \sum_{i=1}^{m} \alpha_i\, a_i\, r\, N(a_i r)}{\displaystyle\int_0^\infty u\,N(u)\,du} + \sum_{i=1}^{m} \alpha_i\,\delta\left(r - \frac{s}{a_i}\right). \tag{4}$$

Here there are m delta-functions with weights $\{\alpha_i\}$, describing the preference of individuals with activity s for individuals with activity s/a_1, s/a_2,..., s/a_m.

For like-with-like preference we might have $a_1 = 1$, and the other a_i arranged as multiples and fractions of unity, with the weights $\{\alpha_i\}$ at a maximum for $i = 1$ and decreasing as the a_i get larger or smaller than unity. It is relatively easy to show that equation (4) satisfies (i) - (iii) provided that the sum of the weights, $\sum_i^m \alpha_i$, is large enough (greater than $\underset{i}{\mathrm{MAX}}\ \{\alpha_i a_i x\, N(x)\}$ is sufficient). It may in practice be a serious deficiency that there are "gaps" in the preference function between the arbitrarily chosen positions of the delta-functions. Nonetheless equation (4) may be useful for preliminary investigations of a like-with-like preference distributed around s = r.

A NEIGHBORHOOD MIXING FUNCTION

Instead of the delta-function model of equation (4), we should like to be free to specify like-with-like preference by some arbitrary function with well-understood properties. In particular we wish to use "neighborhood" functions which express preference as a continuous function with a single peak at r = s, falling off to either side. We know of no such functions which may be used directly, satisfying (i) - (iii). Even an isolated delta-function requires some transformation, the simplest example of which is

$$\rho(s, r) = \frac{r\,N(r)}{s\,N(s)}\,\delta(r - s) \ .$$

This example provides a clue as to how one might make use of an arbitrary function, say $\phi(s,r)$, as our *preference* function. We must ask: "What transformation of the function

$\phi(s,r)$ satisfies (i) - (iii)?" If we restrict our choice of ϕ to functions with the property $\phi(s-r) = \phi(r-s)$, and state that

$$\int_{-\infty}^{+\infty} \phi(y)\, dy = 1 ,$$

then we find that the transformation

$$\rho(s, r) = \frac{r\, N(r)\, P(r)\, P(s)}{\int_0^{\infty} u\, N(u)\, P(u)\, du} + \frac{r\, N(r)}{A} \phi(s - r) , \qquad (5)$$

satisfies (i)-(iii). In (5),

$$P(x) = 1 - \frac{1}{A} \int_0^{\infty} u\, N(u)\, \phi(x - u)\, du , \qquad (6)$$

and A is a constant. We consider this constant further below. It is trivial to show that equation (5) satisfies constraints (ii) and (iii); the value of A must be large enough to give $P(x) > 0$ for all x, which in turn is sufficient to satisfy (i). When $\phi(s-r)$ is not a delta-function at $s = r$, then the choice

$$A = \int_0^{\infty} u\, N(u)\, du \qquad (7)$$

is sufficient; for a delta-function P(x) involves point values rather than integrals, and $A > \text{MAX}\{xN(x)\}$ is necessary and sufficient.

AN EXAMPLE

In this section we consider a simple example for which $\rho(s, r)$ can be calculated. We are not here concerned with a time-varying activity distribution (which would be the case in a real application or a dynamic model), and choose the convenient exponential form

$$N(s) = Nke^{-ks}, \tag{8}$$

where N(s) is the distribution of sexual activity in the population, N is the total population size, and 1/k is the mean sexual activity. For the neighborhood preference functions $\phi(s, r)$, we choose

$$\phi(s, r) = \frac{c}{2} e^{-c|s-r|}, \tag{9}$$

which becomes more sharply peaked as c increases. Using equation (7) we have A = N/k for this case. It is trivial to calculate the expression for P(x) and $\rho(s,r)$ given equations (8) and (9), and in Figures (1) to (10) we present some illustrative examples. In the figures, we have graphed $\rho(s,r)$ as a function of r for different values of s and for a variety of values of c and k, with A = N/k.

In Figs (1) to (3) we illustrate $\rho(s,r)$ for k = 0.1 and c = 0.5, and s = 1.0, 5.0, and10.0 respectively. In this case $\rho(s,r)$ retains the sharply peaked form of $\phi(s,r)$ except when s is small, in which case $\rho(s,r)$ is much smoother. This case corresponds to a very narrow neighborhood function, with 50% of the area under $\phi(s,r)$ lying in the interval $r = s \pm 2\ln 2$, and a large average activity: 1/k = 10.0 partners per unit time.

In Figs (4) to (6) we illustrate $\rho(s,r)$ for k = 0.5 and c = 0.1, with the same range of s values. In this case the neighborhood function is very broad, and contributes very little to the shape of $\rho(s,r)$, which always behaves as rN(r) (that is, like proportionate mixing).

In Figs (7) to (10) we illustrate the case $k = 0.25$ and $c = 1.0$ for $s = 1.0$, 5.0 10.0, and 20.0, respectively. Although here the neighborhood function is narrow, the mean sexual activity is small and the interplay between $N(r)$ and $\rho(s,r)$ is complicated. The essential form of $\rho(s,r)$ is a mixture of proportionate and like-with-like mixing. At a small s (less than 1/k, Fig (7)), $\rho(s,r)$ is very much like $\phi(s,r)$, but with a more pronounced tail. As s is increases (Figs (8) to (10)), the component due to $\phi(s,r)$ decreases, until by the time $s = 20.0$ proportionate mixing is predominant.

We remark that the fidelity of the transformation $\rho(s,r)$ to the underlying neighborhood function $\phi(s,r)$, given equations (8) and (9), depends upon the width of ϕ, the mean activty 1/k, and the value of s in relation to 1/k.

CONCLUSION

We have presented two new like-with-like mixing functions, one based on proportionate mixing biased at m values of the ratio s/r, and the other based on a transformation of a general neighborhood function $\phi(s,r)$. A simple example for a static population indicates that the second mixing function behaves like the neighborhood function, provided that the latter is sharply peaked and the mean activity in the population is relatively high. In other cases proportionate mixing may be regained, with or without a level of bias towards like-with-like preference. These results support some of the numerical experiments of Hyman and Stanley (1988, and in press) regarding the role of the width (variance) of the neighborhood preference function and its relationship to proportionate mixing.

Much work remains to be performed before we have a complete understanding of the transformation method for an arbitrary neighborhood function, and the behavior of this $\rho(s,r)$ in a fully dynamic epidemiological model must be investigated. This work is in progress and we hope to report on it in future publications. Finally, we speculate that if estimates for N(s) (the activity distribution in the population) and $\phi(s,r)$ (tendency for like-with-like mixing) can be obtained from survey results, then examination of the transformation $\rho(s,r)$ of equation (5) may be able to tell us whether or not the like-with-like preference is important in a given population, and thus whether a proportionate mixing description is adequate, or a more complicated model is required.

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FIGURE CAPTIONS

Fig 1. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.1$, $c = 0.5$, and $s = 1.0$.

Fig 2. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.1$, $c = 0.5$, and $s = 5.0$.

Fig 3. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.1$, $c = 0.5$, and $s = 10.0$.

Fig 4. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.5$, $c = 0.1$, and $s = 1.0$.

Fig 5. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.5$, $c = 0.1$, and $s = 5.0$.

Fig 6. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.5$, $c = 0.1$, and $s = 10.0$.

Fig 7. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.25$, $c = 1.0$ and $s = 1.0$.

Fig 8. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.25$, $c = 1.0$, and $s = 5.0$.

Fig 9. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.25$, $c = 1.0$, and $s = 10.0$.

Fig 10. Behavior of (a) $\phi(s,r)$ and (b) $\rho(s,r)$ for $k = 0.25$, $c = 1.0$, and $s = 10.0$.

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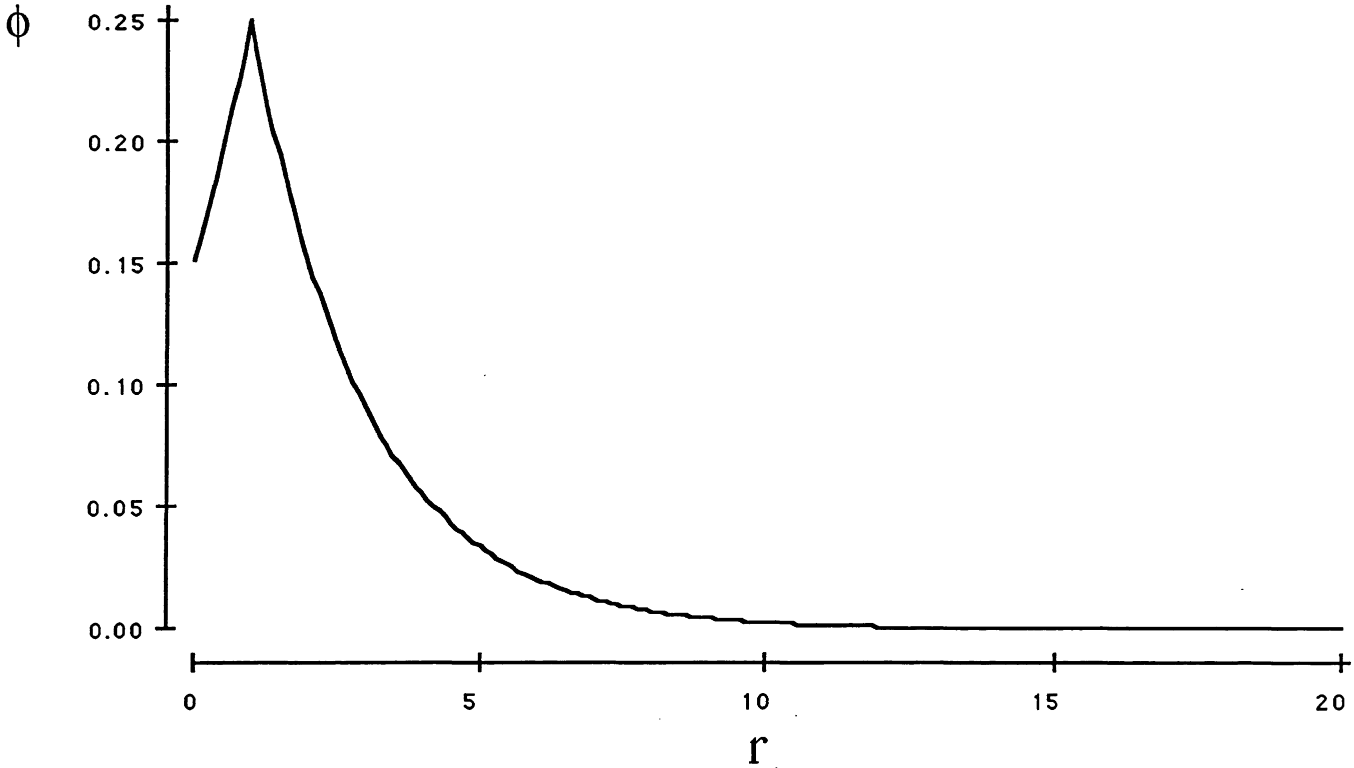

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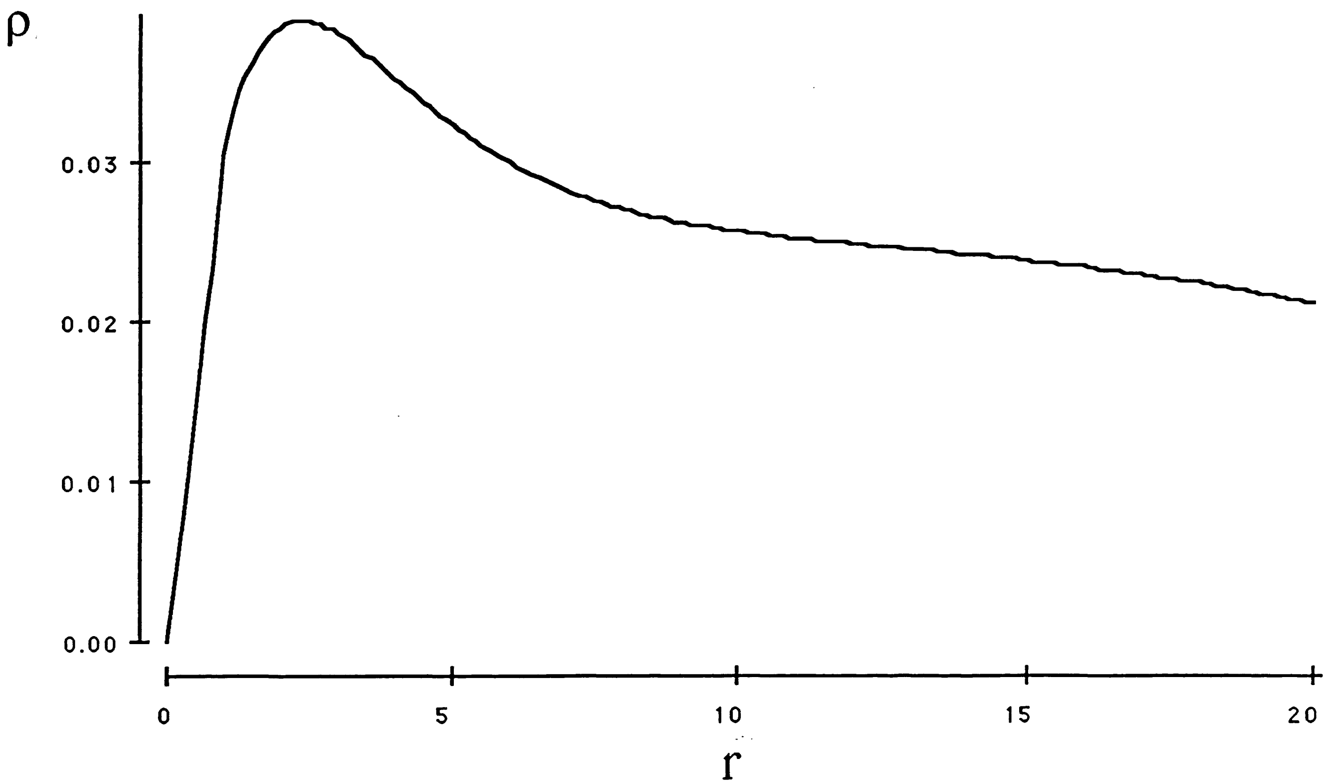

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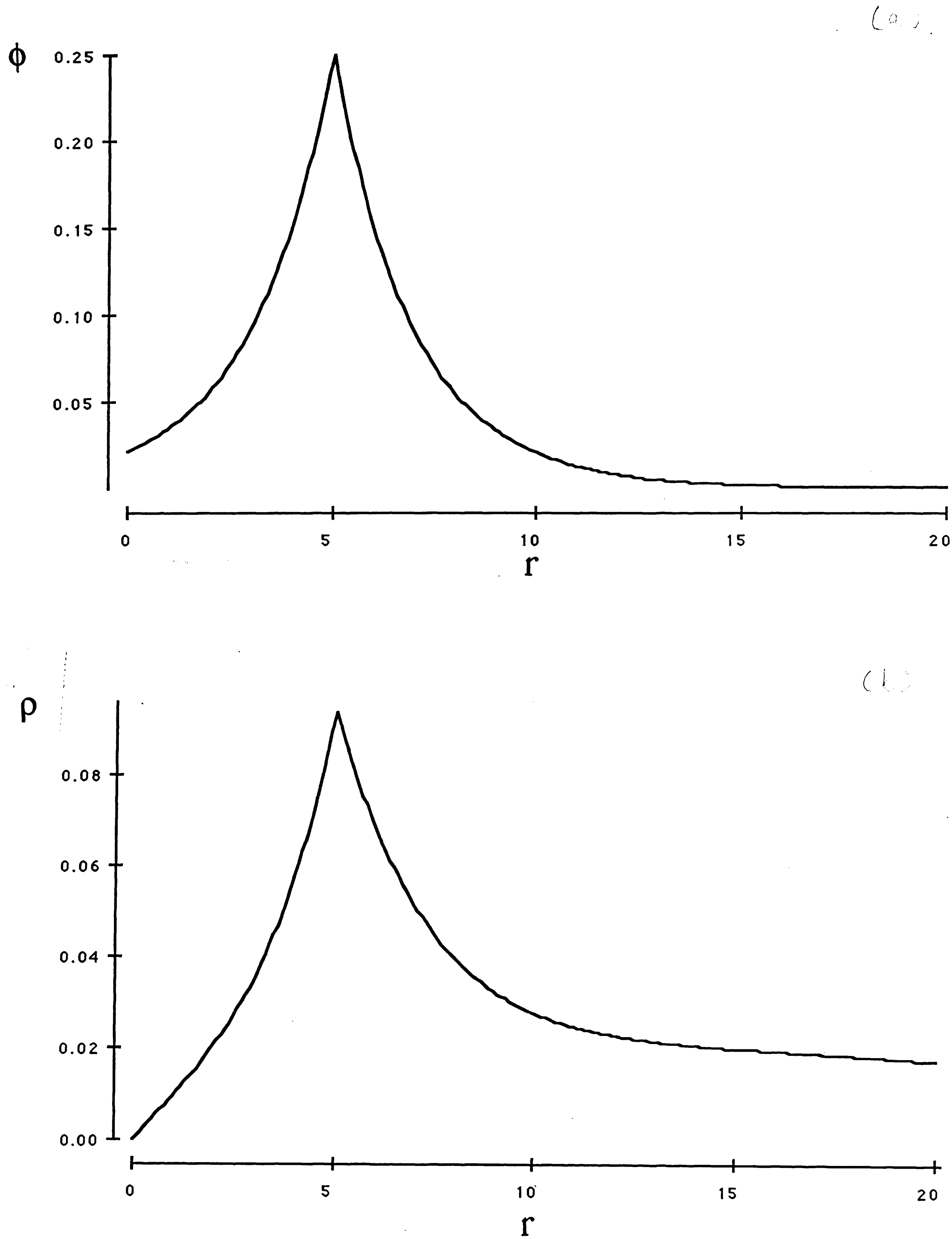
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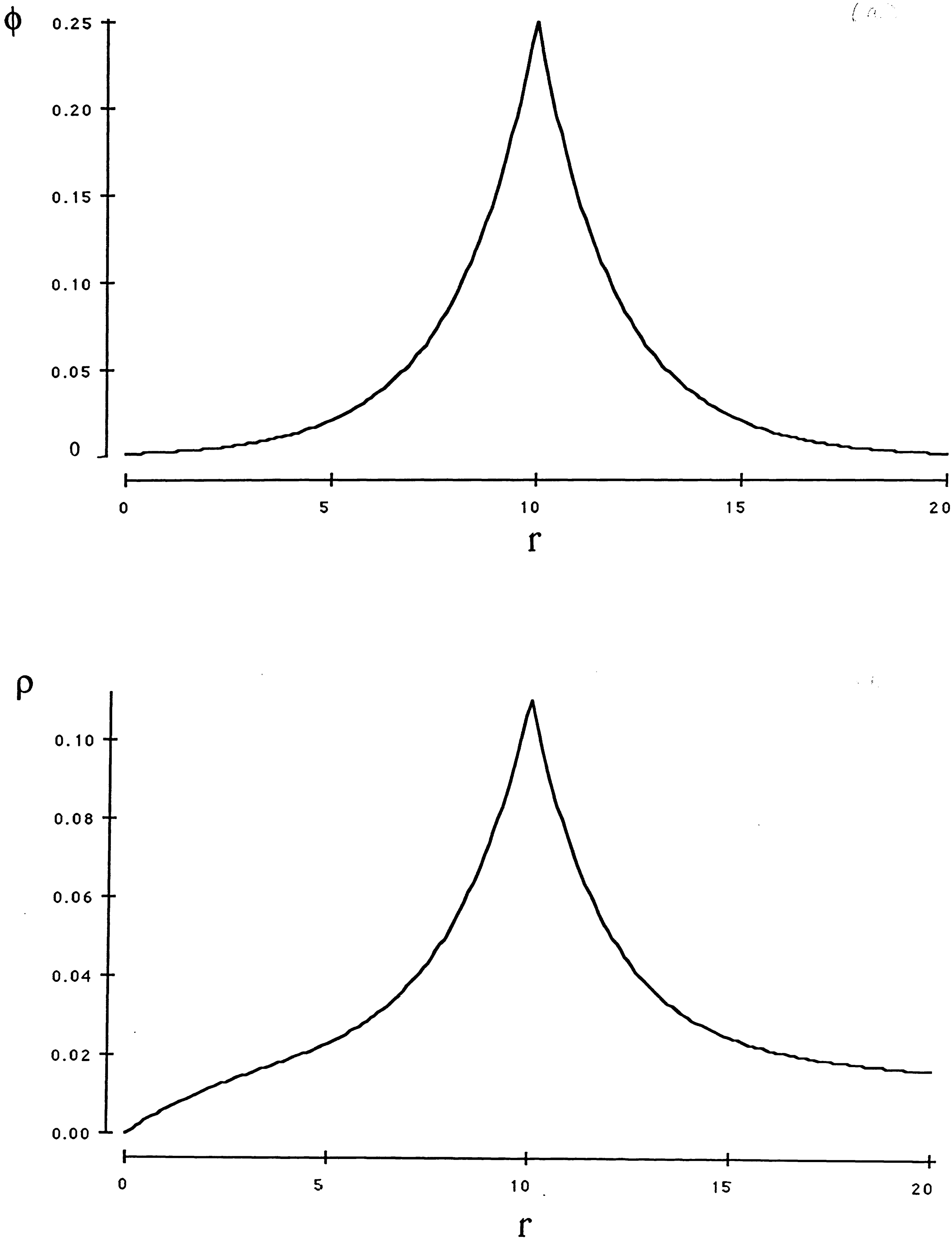

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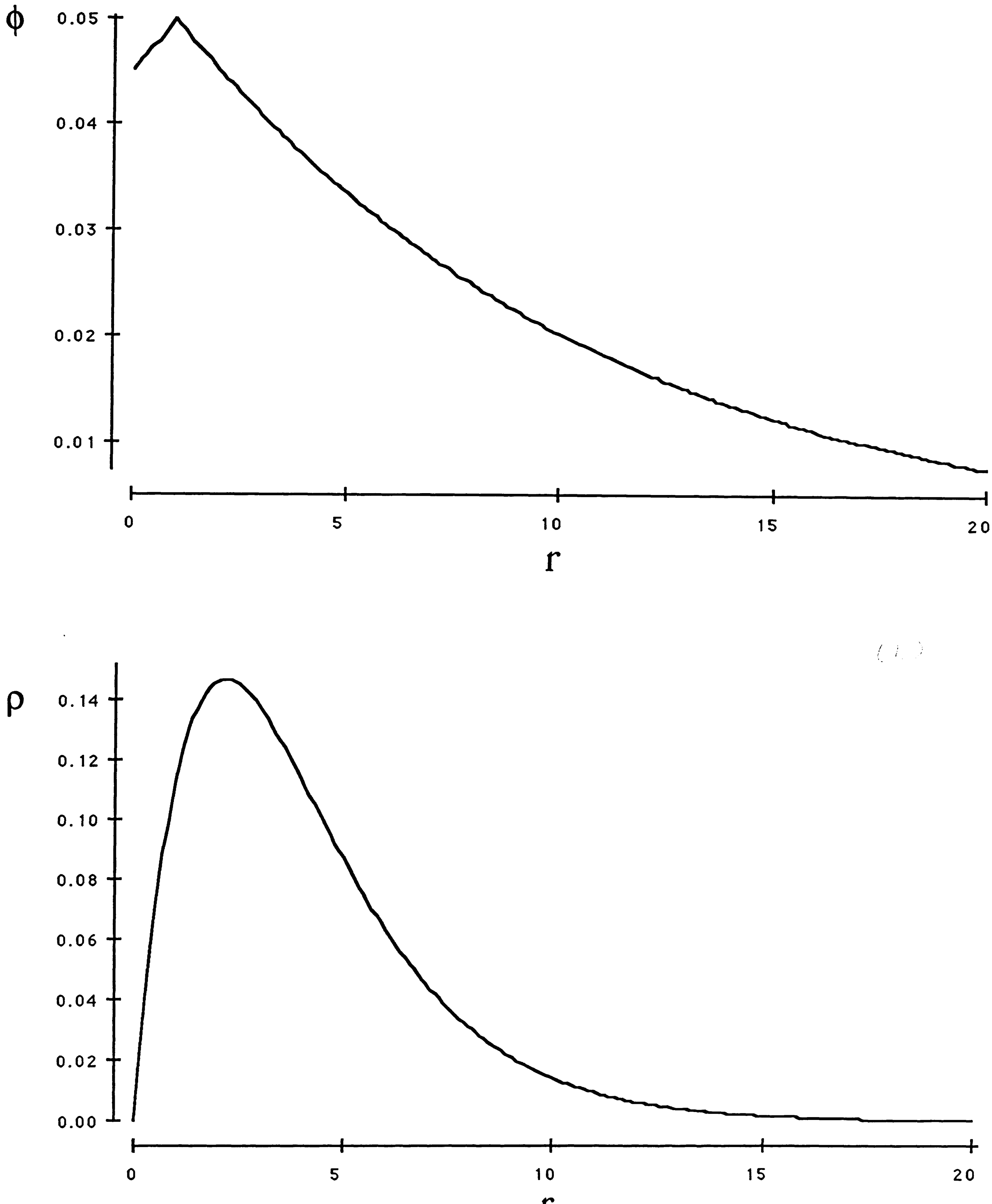
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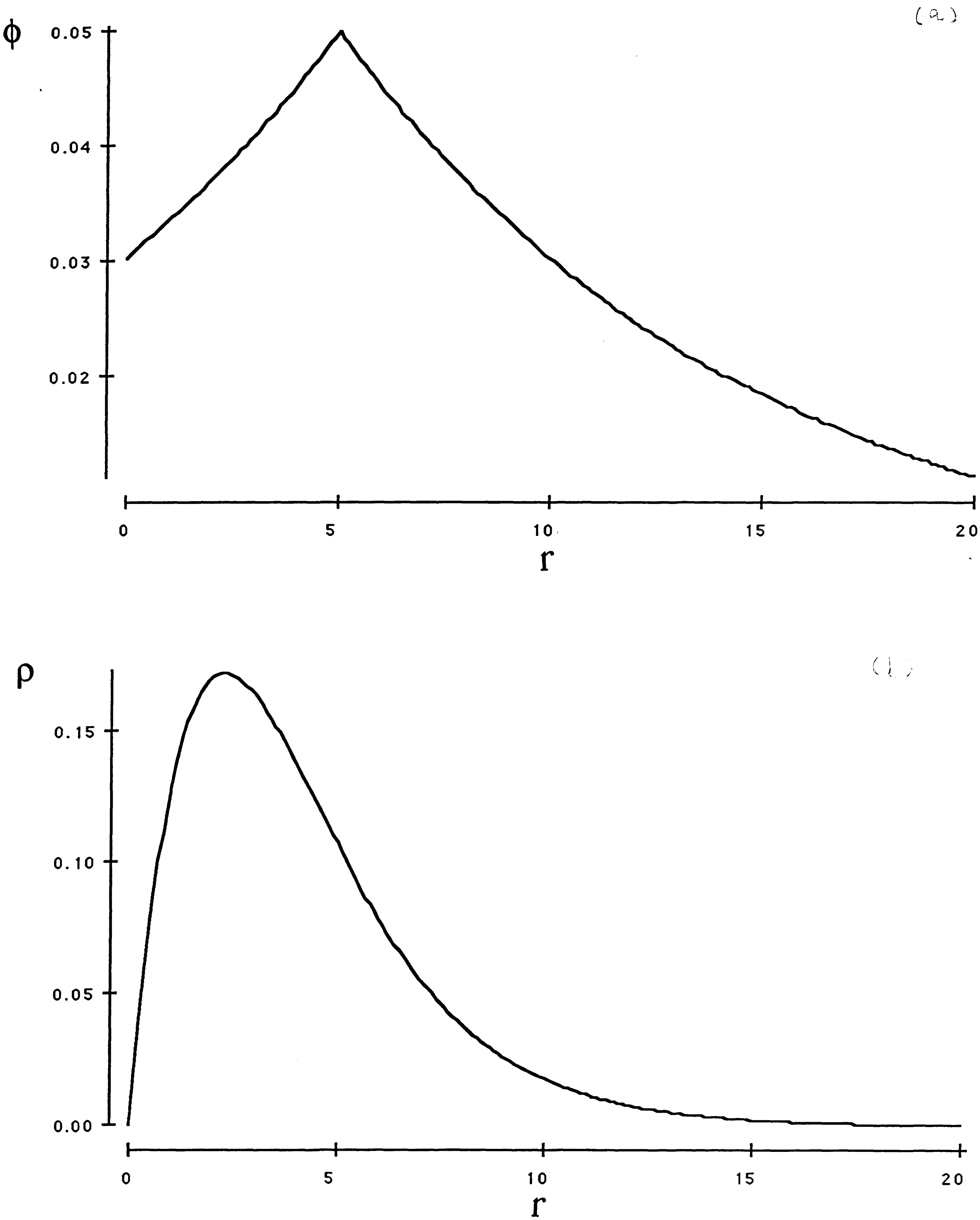

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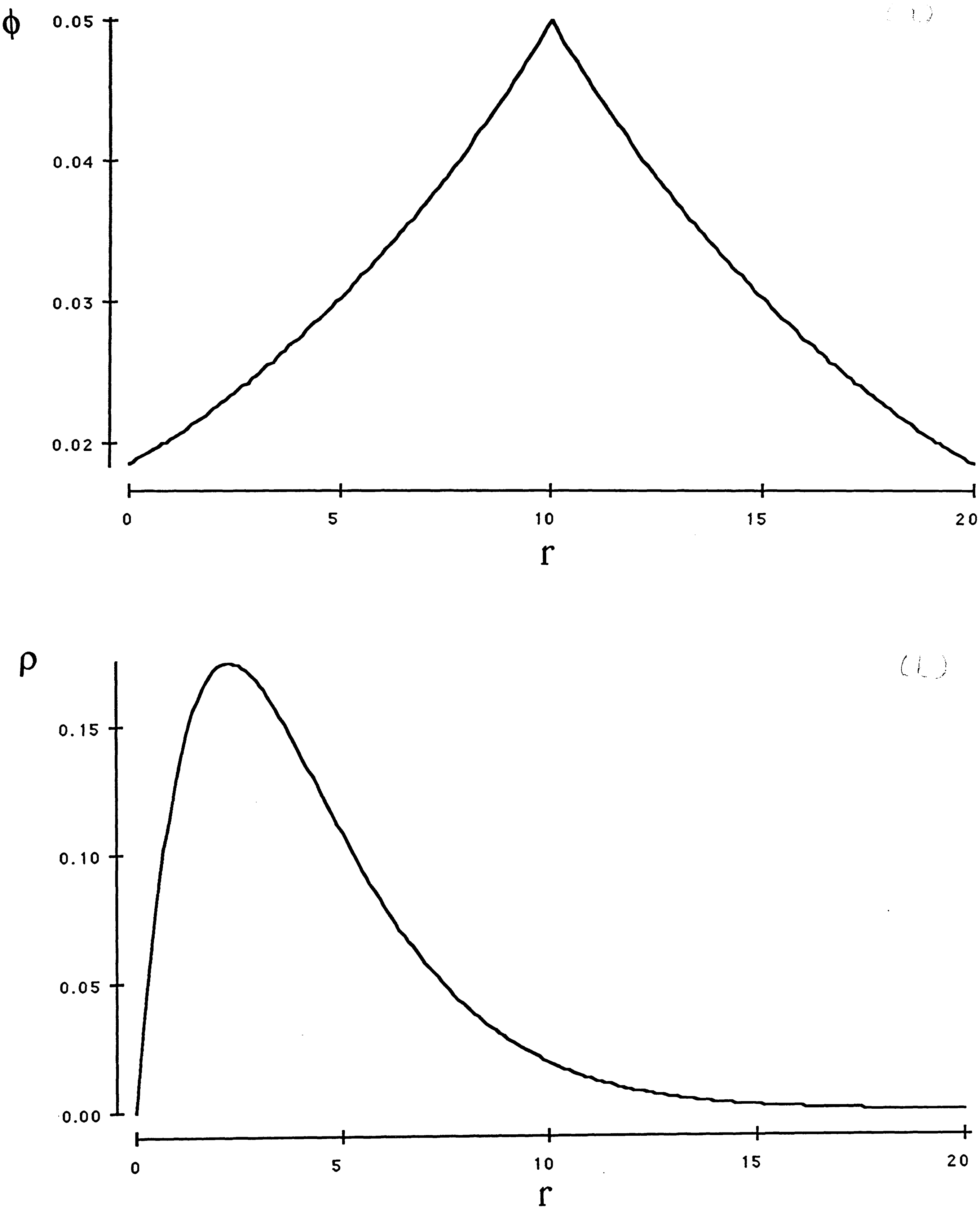

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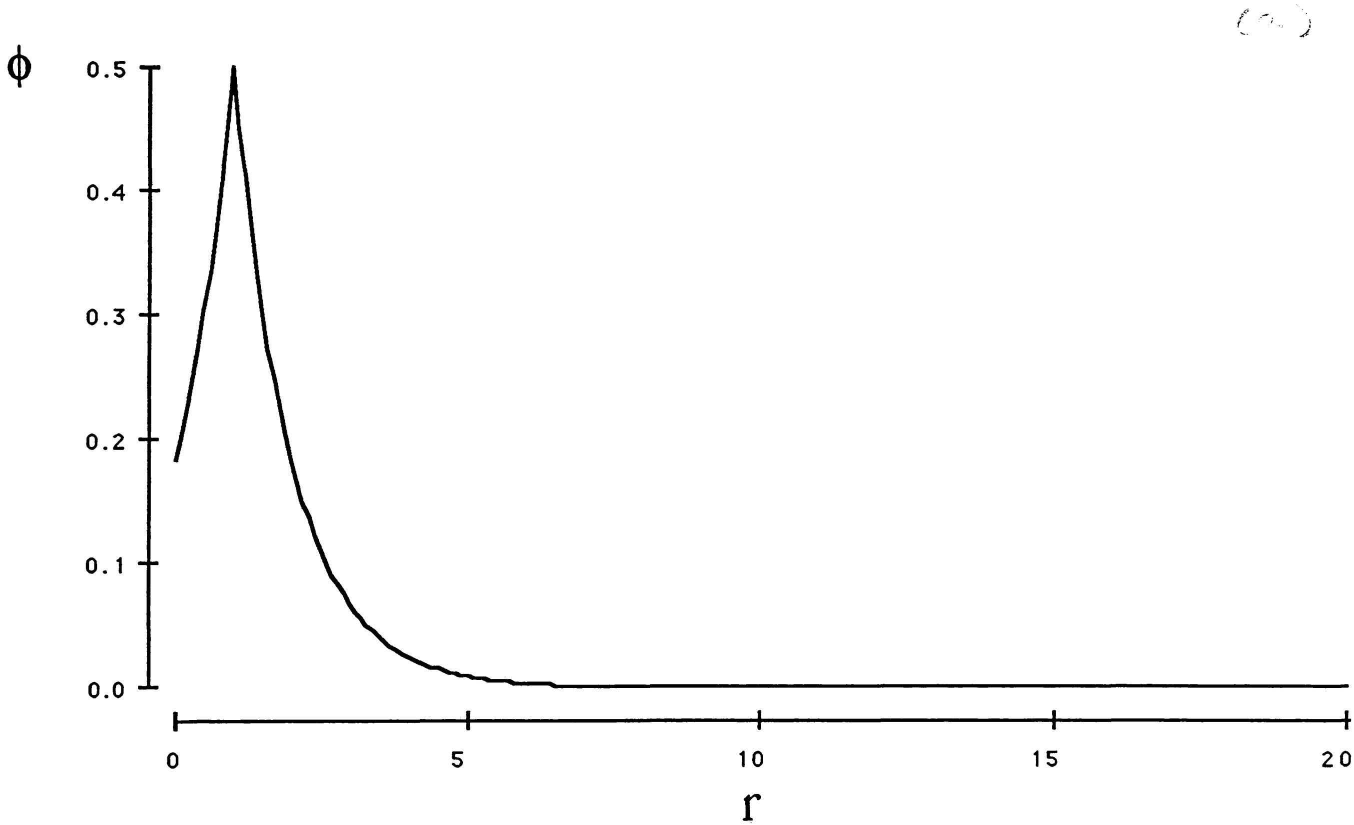
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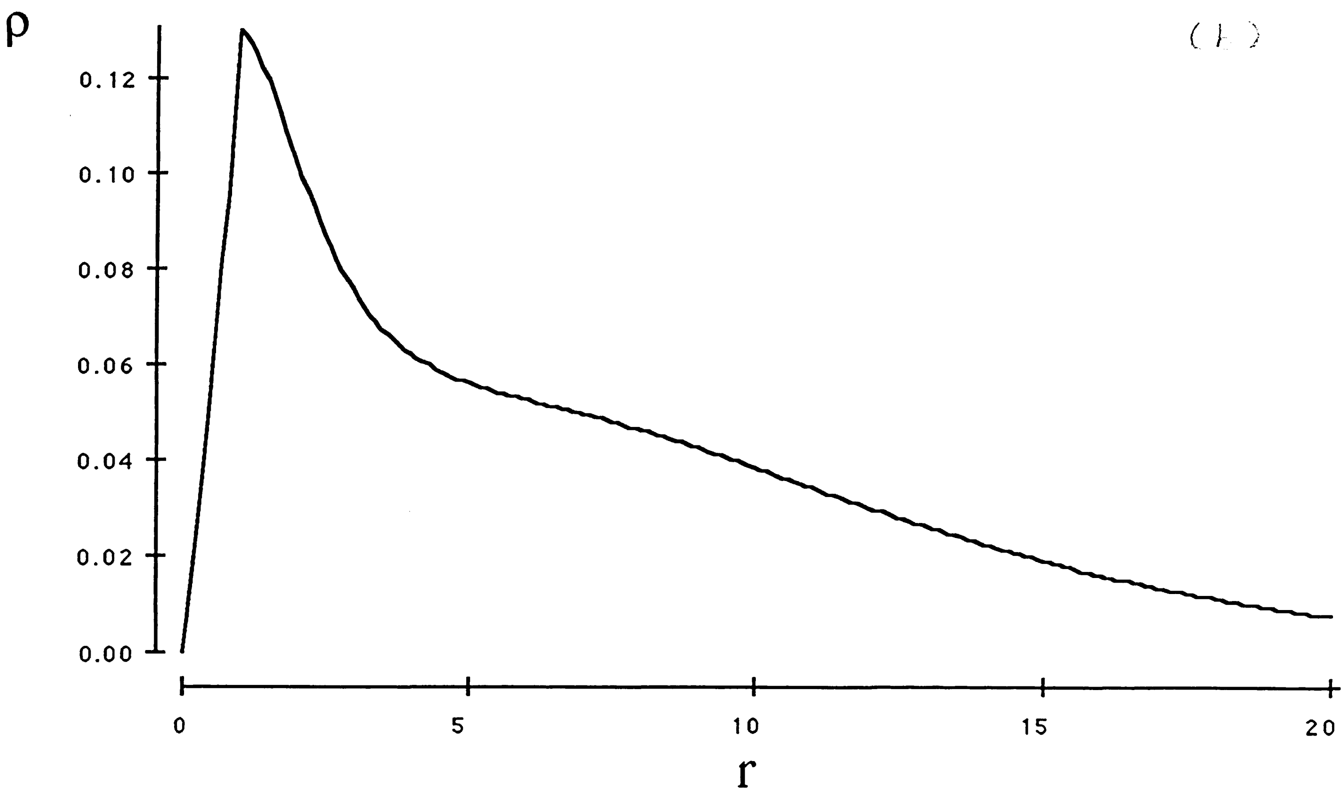
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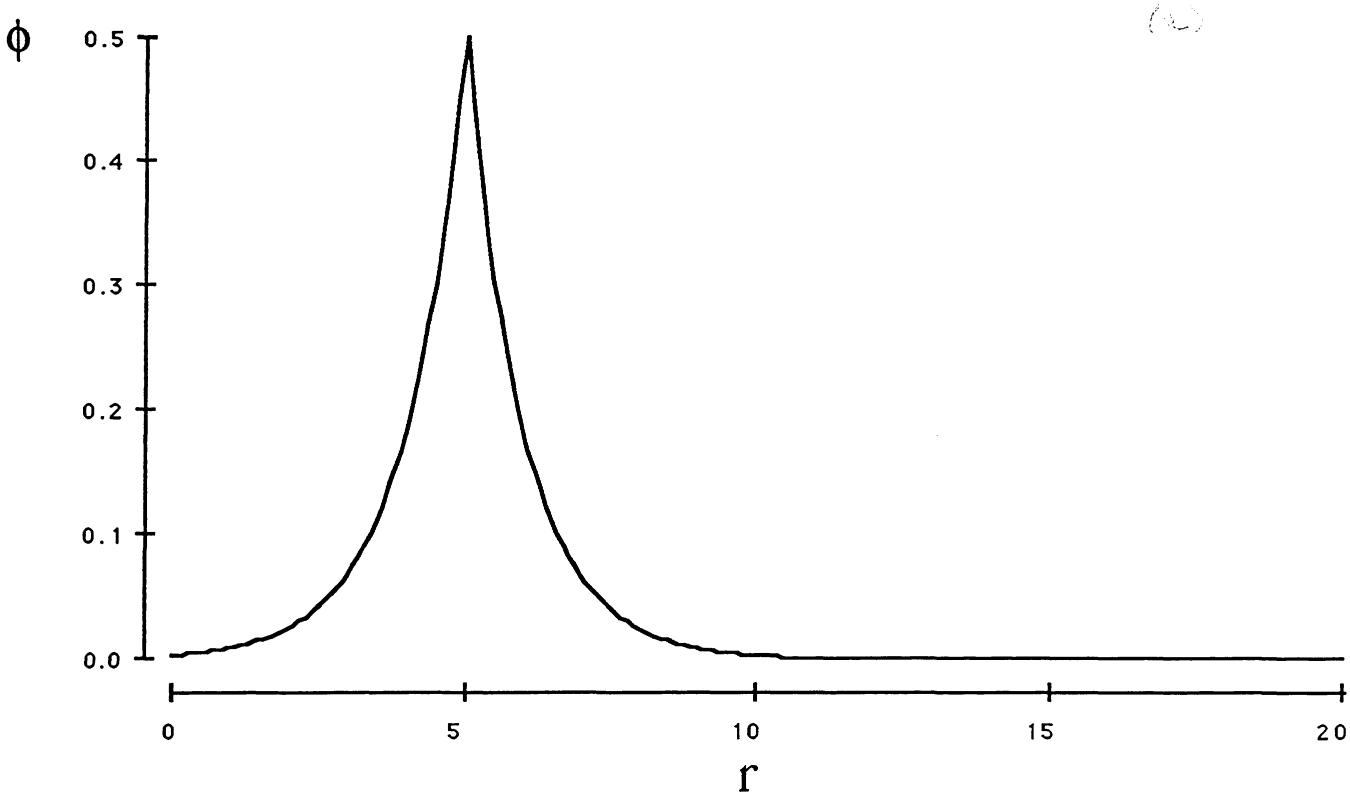

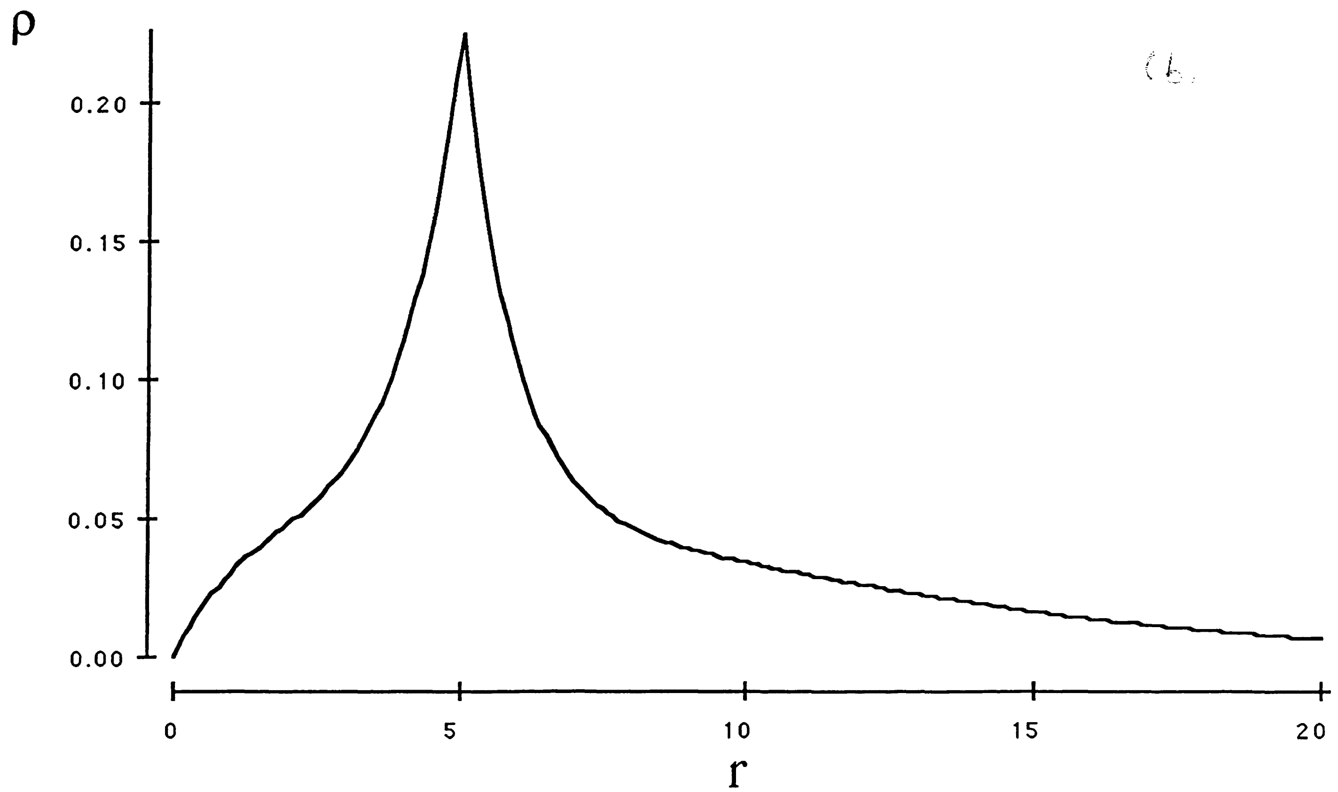

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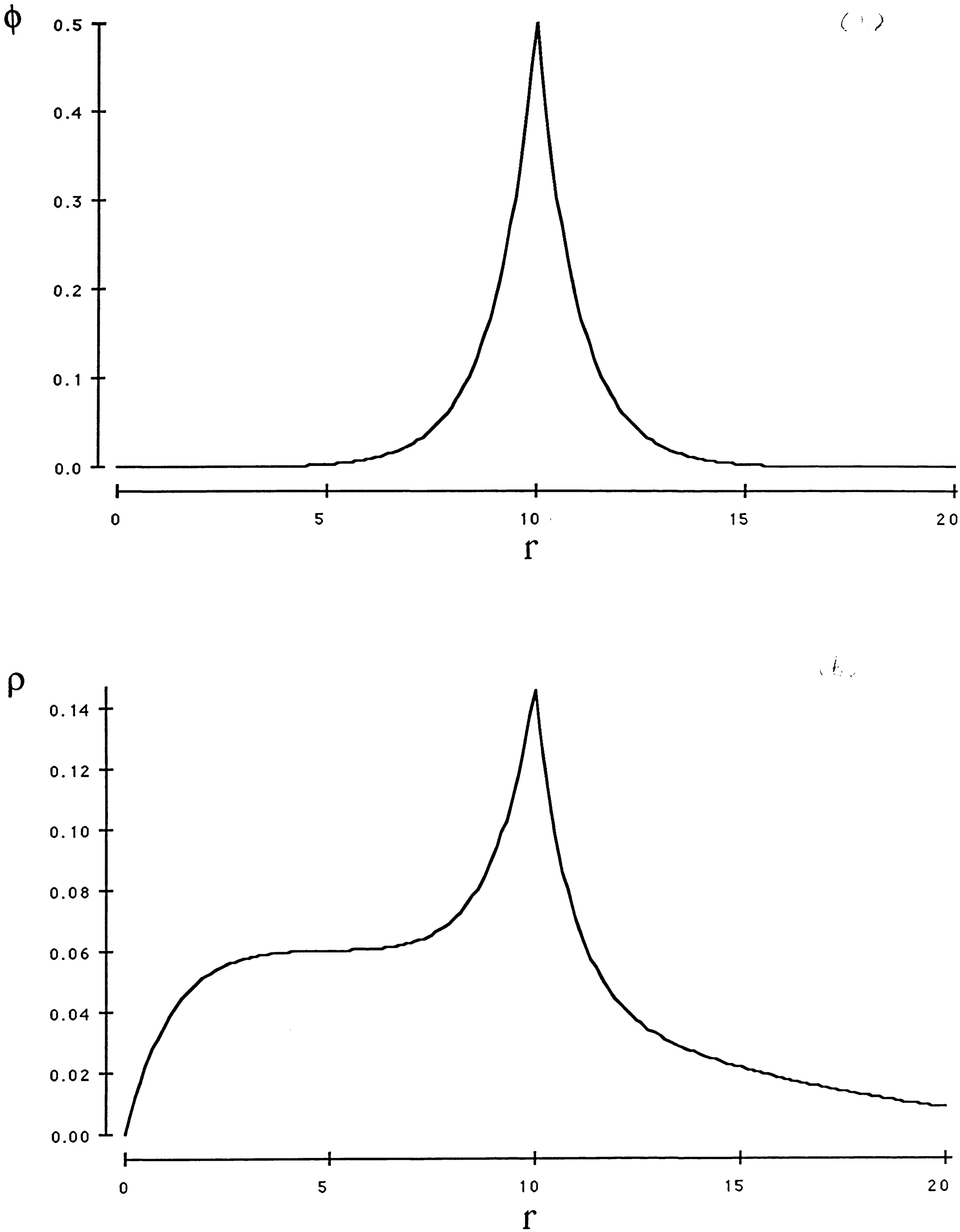
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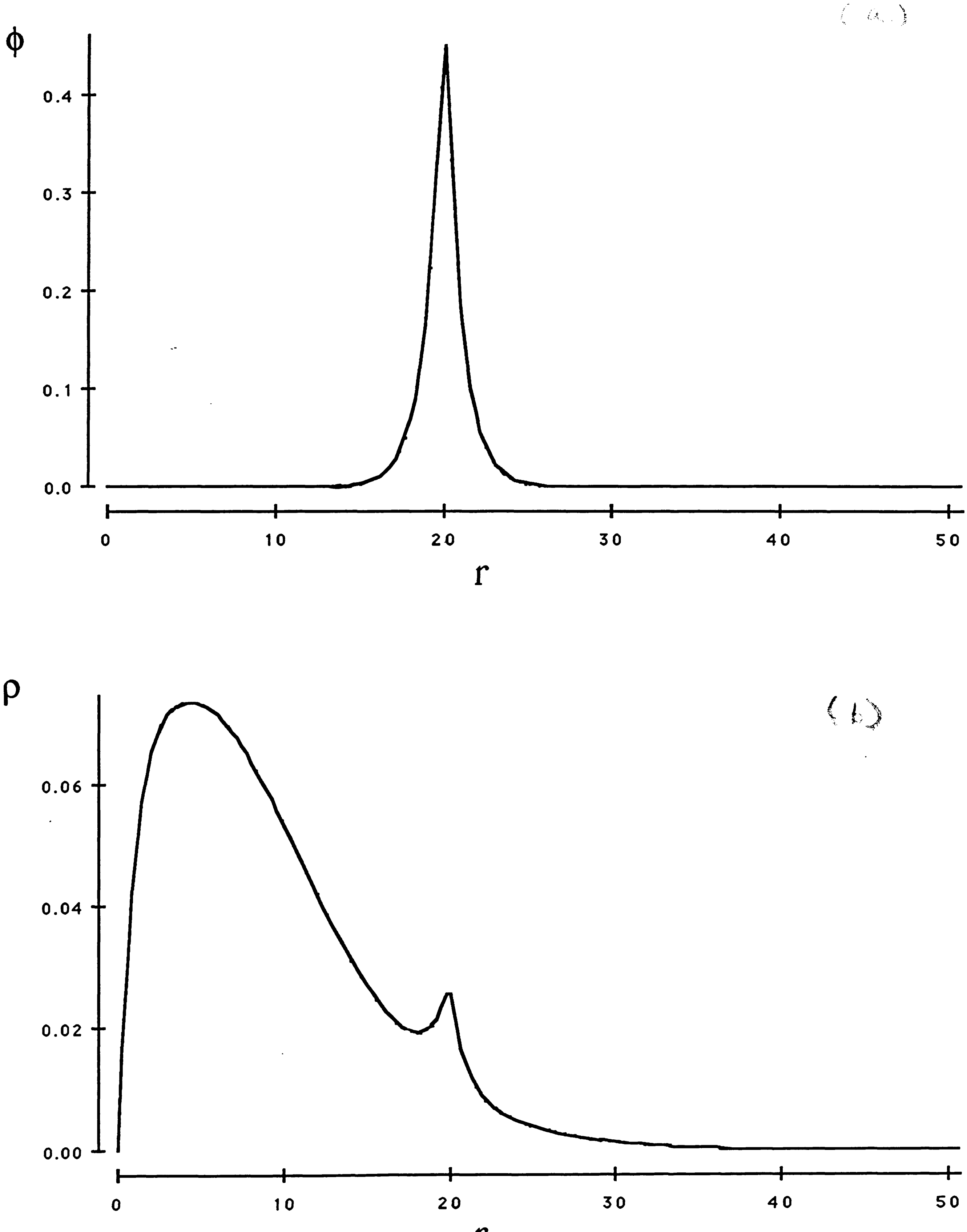
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Fig 10